B.Sc-part-II(H), paper-III Tangents and normals. 23-07-2020

Problems based on the angle of intersection of polar curves

Q.1 Find the angle of intersection of the cardioids $r = a(1+\cos\theta)$, $r = b(1-\cos\theta)$.

Soln: The first curve is $r = a(1+\cos\theta)$

Diff. w.r.t. θ, $\frac{dr}{d\theta} = -a\sin\theta$

We know that $\tan\phi = r\frac{d\theta}{dr} = \frac{a(1+\cos\theta)}{-a\sin\theta} = \frac{\cancel{2}\cos^{\cancel{2}}\frac{\theta}{2}}{-\cancel{2}\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$

$= -\cot\frac{\theta}{2} = \tan\left(\frac{\pi}{2}+\frac{\theta}{2}\right)$

$\therefore \phi = \frac{\pi}{2}+\frac{\theta}{2}$

The second curve is $r = b(1-\cos\theta)$

Diff w.r.t. to θ, $\frac{dr}{d\theta} = b\sin\theta$

We have, $\tan\phi' = r\frac{d\theta}{dr} = \frac{\cancel{b}(1-\cos\theta)}{\cancel{b}\sin\theta} = \frac{\cancel{2}\sin^{\cancel{2}}\frac{\theta}{2}}{\cancel{2}\sin\frac{\theta}{2}\cancel{\cos\frac{\theta}{2}}} = \tan\frac{\theta}{2}$

$\therefore \tan\phi' = \tan\frac{\theta}{2}$

$\therefore \phi' = \frac{\theta}{2}$

The required angle of intersection

$= \phi - \phi' = \frac{\pi}{2}+\cancel{\frac{\theta}{2}}-\cancel{\frac{\theta}{2}} = \frac{\pi}{2}$

Thus we find that the two given curves cut each other orthogonally.

Q.2. For the cardioide $r = a(1-\cos\theta)$, Prove the following results (i) $\phi = \frac{\theta}{2}$ (ii) $p = 2a\sin^3\frac{\theta}{2}$ (iii) $p^2 = \frac{r^3}{2a}$

Soln: We know that $p = r\sin\phi$ —— (i)

$\tan\phi = r\frac{d\theta}{dr}$ —— (ii)

$\because r = a(1-\cos\theta)$, $\therefore \frac{dr}{d\theta} = a\sin\theta$

Also, $r = 2a\sin^2\frac{\theta}{2}$

$\therefore \tan\phi = \cancel{2a}\cancel{\sin^2\frac{\theta}{2}}\cdot\frac{1}{2a\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = \tan\frac{\theta}{2}$ $\therefore \phi = \frac{\theta}{2}$

(ii) Again, $p = r\sin\phi = 2a\sin^2\frac{\theta}{2}\cdot\sin\frac{\theta}{2} = 2a\sin^3\frac{\theta}{2}$

Also, (iii) $p^2 = r^2\sin^2\phi = r^2\sin^2\frac{\theta}{2}$ (as $\phi = \frac{\theta}{2}$) $= r^2\cdot\frac{r}{2a} = \frac{r^3}{2a}$

Proved the following results.

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Q.3. In the parabola $\frac{2a}{r} = 1 - \cos\theta$, prove the following results.

(i) $\phi = \pi - \frac{\theta}{2}$ (ii) $p = \frac{a}{\sin\frac{\theta}{2}}$ (iii) $p^2 = ar$

(iv) polar sub-tangent $= 2a\operatorname{cosec}\theta$.

Soln: We know that $p = r\sin\phi$ ——— (1)

$\tan\phi = r\frac{d\theta}{dr}$ ——— (2)

$\because \frac{2a}{r} = 1 - \cos\theta = 2\sin^2\frac{\theta}{2}$

$\therefore 2a\left(-\frac{1}{r^2}\right)\frac{dr}{d\theta} = \sin\theta = 2\sin\frac{\theta}{2}\cdot\cos\frac{\theta}{2}$

(i) $\therefore \tan\phi = r\frac{d\theta}{dr} = \frac{a}{\sin^2\frac{\theta}{2}} \times \frac{(-a)}{r^2\, \sin\frac{\theta}{2}\cos\frac{\theta}{2}}$

$= \frac{a}{\sin^2\frac{\theta}{2}} \cdot \frac{\sin^4\frac{\theta}{2}}{a^2} \times \frac{(-a)}{\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$

$= -\cancel{\tan\theta} = \tan\left(\pi - \frac{\theta}{2}\right) \quad \therefore \phi = \pi - \frac{\theta}{2}$

II Again $p = r\sin\phi = \frac{a}{\sin^2\frac{\theta}{2}} \times \sin\left(\pi - \frac{\theta}{2}\right) = \frac{a}{\sin\frac{\theta}{2}}$

III Also, $p^2 = r^2\sin^2\phi = \cancel{\frac{a}{\sin\frac{\theta}{2}} \times \sin\left(\pi - \frac{\theta}{2}\right) = \frac{a}{}}$

$= r^2\sin^2\left(\pi - \frac{\theta}{2}\right) = r^2\sin^2\frac{\theta}{2} = r^2 \cdot \frac{a}{r} = ar$

(iv) Now, polar sub-tangent $= r^2\frac{d\theta}{dr} = \frac{2a}{\sin\theta} = 2a\operatorname{cosec}\theta$

Which is the required results.

Proved.

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